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MACROMECHANICAL PARAMETRIC AMPLIFICATION WITH A BASE-EXCITED DOUBLY CLAMPED BEAM

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Abstract. *Parametric amplification is realized by adding parametric excitation to externally driven near-resonant oscillations. The effect of specific cubic nonlinearities on the parametrically amplified steady-state vibrations and gain is investigated theoretically. Here, gain is defined as the ratio of steady-state vibration amplitude of the directly and parametrically excited system, to vibration amplitude of the directly excited only system. The nonlinear effect of midplane stretching is compared to the effects of nonlinear inertia and curvature. An approximate analytical expression for the vibration amplitude is derived. For a given small level of transverse displacement for both the cantilever and doubly clamped beam, the effect of midplane stretching is dominant compared to those caused by nonlinear inertia and curvature. It was found that the beam slenderness ratio can be used as an effective design parameter for parametric amplifiers.*

1 INTRODUCTION

Parametric excitation appears as time-dependent coefficients in the governing equations. These coefficients are related to terms associated with stiffness and/or mass [1, 2]. Parametric *pumping*, in this context adding parametric excitation to externally driven near-resonant oscillations, gives *parametric amplification* as long as the parametric excitation is below the instability threshold above which parametric resonance occurs. Parametric pumping can occur for a cantilever beam at twice the resonant frequency [3]. This 2:1 relationship between parametric and direct excitation is the simplest parametric amplification scheme [4], referred to as *perfectly tuned* or *degenerate*.

Many linear parametric amplifiers exhibit a narrow bandwidth [5–8]. Recent studies [9–13] have focused on increasing the gain bandwidth. Here *gain* is defined as the ratio between the steady-state vibration amplitudes of the pumped and unpumped system. Common approaches for increasing the operating bandwidth include techniques such as frequency up-conversion, resonance tuning and utilizing nonlinearities: In [9], the nonlinear stiffness of a doubly clamped beam was considered, showing that the bandwidth of an energy harvester broadened, thus making the vibration amplitude less sensitive to a mismatch between the excitation and resonant frequency. This is advantageous for e.g. energy harvesters, because the ambient motion and vibration may vary with environment. However [14] reports that nonlinear effects may reduce the steady-state vibration amplitude for perfectly tuned parametric amplifiers. Realization of parametric amplification in a macroscale mechanical context was demonstrated in [15], with subsequent consideration to nonlinear effects on the gain in [16].

This work examines the effect of commonly occurring cubic nonlinearities on the parametrically amplified steady-state vibration amplitude response and gain. Specifically, we consider the effects of midplane stretching and nonlinear inertia and curvature, relevant with doubly clamped beams and cantilever beams, respectively. These nonlinearities can yield qualitative and quantitative different results, depending on e.g. the beam slenderness ratio. In particular, for similar transverse displacements, the nonlinearity of a doubly clamped beam is much stronger than for a cantilever beam. Thus, nonlinear effects might be easier utilized (or give more challenges) for doubly clamped beams. The findings indicate that common nonlinearities might find application for parametric amplification purposes in mechanics.

2 MODEL SYSTEM

A base-excited doubly clamped beam is considered as a representative model system. We assume that nonlinearities are weak, the beam is slender and elastic, shear deformations, longitudinal and rotatory inertia and gravity can be neglected, and cross section rotations and damping are small. Parametric amplification is obtained by tilting the doubly clamped beam with respect to the line of excitation x as shown in Figure 1(b); this arrangement was recently realized in experimental laboratory setups for investigating macromechanical parametric amplification for cantilever beams (Figure 1(a), [15, 17, 18]). The imposed base motion x_b , with axial and transverse components u_b and v_b , is provided by a vibration exciter, and the tilt angle α relates the beam axis to the line of excitation. Using Hamiltons extended principle one obtains a nonlinear partial differential equation which governs the longitudinal $u(s, t)$ and transverse $v(s, t)$ beam displacements (with respect to the moving base):

$$\ddot{v} + \frac{c}{\rho A} \dot{v} + \frac{EI}{\rho A} v'''' - \left(\frac{1}{2l} \frac{E}{\rho} \int_0^l (v')^2 ds + (s - l) \ddot{u}_b \right) v'' - (\ddot{u} + \ddot{u}_b) v' = -\ddot{v}_b, \quad (1)$$

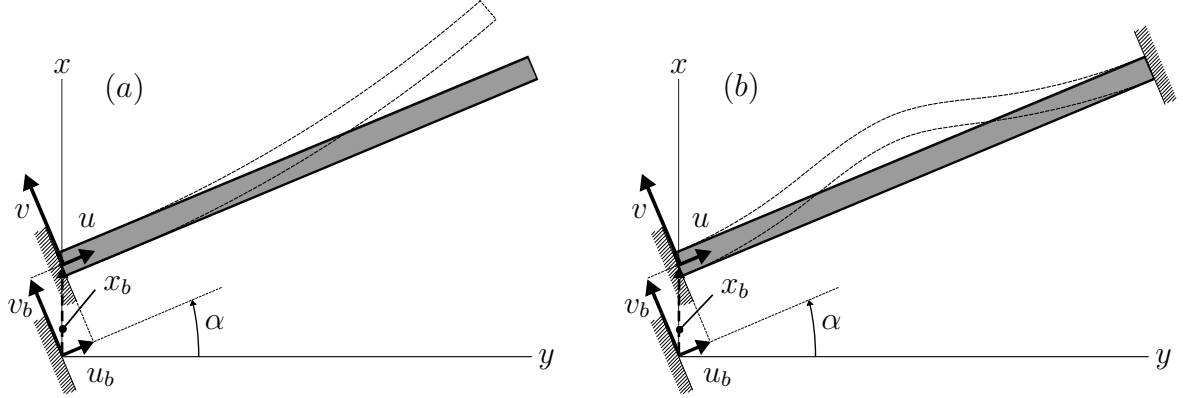


Figure 1: Base-excited tilted: (a) cantilever beam; (b) doubly clamped beam. Inertial reference coordinates x and y . Imposed base motion with displacement components \hat{u}_b and \hat{v}_b , at tilt angle α . Longitudinal $\hat{u}(s, t)$ and transverse $\hat{v}(s, t)$ beam displacements.

where $s \in [0; l]$ is the axial coordinate, l the beam length, t is time, $(\dot{})$ and $()'$ denote temporal and spatial derivatives, c is the damping coefficient, ρ the density of the beam, A the cross-sectional area, E the elastic modulus, and I the area moment of inertia. The axial inertia is considered negligible compared to the transverse inertia, and therefore omitted in subsequent analyses. Introducing nondimensional variables $\hat{t} = t/T$, $\hat{s} = s/l$, $\hat{c} = cT/\rho A$, $T = \sqrt{\rho A l^4 / EI}$, $r = \sqrt{I/A}$, $\lambda = l/r$, where T is a characteristic time, r the radius of gyration of the cross-section, and λ the beam slenderness ratio, into (1), yields corresponding nondimensional system:

$$\ddot{\hat{v}} + \hat{c}\dot{\hat{v}} + \hat{v}'''' - \left(\frac{1}{2}\lambda^2 \int_0^1 (\hat{v}')^2 d\hat{s} + (\hat{s} - 1)\ddot{\hat{u}}_b \right) \hat{v}'' + \ddot{\hat{u}}_b \hat{v}' = -\ddot{\hat{v}}_b, \quad (2)$$

where $\hat{v} = \hat{v}(\hat{s}, \hat{t})$. The base displacement \hat{x}_b is assumed to be two-frequency time-harmonic:

$$\hat{x}_b = \hat{A} \cos(\hat{\omega}\hat{t} + \phi) + \hat{B} \cos(2\hat{\omega}\hat{t}), \quad (3)$$

with the components:

$$\hat{u}_b = \hat{x}_b \sin \alpha, \quad \hat{v}_b = \hat{x}_b \cos \alpha. \quad (4)$$

Thus, the *direct amplitude* \hat{A} quantifies the part of the shaker input supposed to excite the lowest beam resonance directly, while the *pumping amplitude* \hat{B} quantifies the shaker input exciting the beam at primary parametric resonance, i.e. at twice a natural frequency, and ϕ is the phase between the parametric and direct excitation. A tilt angle $\alpha = 0$ refers to a positioning of the doubly clamped beam where pure external excitation occurs, while for $\alpha = \pm\pi/2$, the excitation is purely parametric. Assuming a single-mode approximation $\hat{v}(\hat{s}, \hat{t}) = w(\hat{t})\Phi(\hat{s})$, where $\Phi(\hat{s})$ is the fundamental mode shape, one obtains a ordinary differential equation for the case of perfect external and parametric tuning:

$$\begin{aligned} \ddot{w} + 2\varepsilon\zeta\dot{w} + (1 + \varepsilon\beta_1\Omega^2 \cos(\Omega\tau + \phi) + \varepsilon\beta_2\Omega^2 \cos(2\Omega\tau)) w + \varepsilon\kappa_4 w^3 \\ = \varepsilon\eta_1\Omega^2 \cos(\Omega\tau + \phi) + \varepsilon\eta_2\Omega^2 \cos(2\Omega\tau), \end{aligned} \quad (5)$$

where $\beta_1, \beta_2, \kappa_4, \eta_1$ and η_2 are defined in Table 1. Here ε bookmark terms assumed to be small, ζ is the damping ratio, Ω the normalized excitation frequency, and τ the time.

$\varepsilon\eta_1 = \hat{A} \cos(\alpha) \int_0^1 \Phi d\hat{s},$	$\varepsilon\eta_2 = 4\hat{B} \cos(\alpha) \int_0^1 \Phi d\hat{s}$
$\varepsilon\beta_1 = \hat{A} \sin(\alpha) \int_0^1 \Phi'' \Phi (\hat{s} - 1) d\hat{s},$	$\varepsilon\beta_2 = 4\hat{B} \sin(\alpha) \int_0^1 \Phi'' \Phi (\hat{s} - 1) d\hat{s}$
$\varepsilon\omega^2 \kappa_4 = -\frac{1}{2} \lambda^2 \int_0^1 \Phi'' \Phi d\hat{s} \int_0^1 (\Phi')^2 d\hat{s},$	$\omega^2 = \int_0^1 \Phi'''' \Phi d\hat{s}, \quad \int_0^1 \Phi^2 d\hat{s} = 1$
$\Omega = \hat{\omega}/\omega, \quad \varepsilon\zeta = \hat{c}/2\omega,$	$\tau = \omega\hat{t}, \quad (\dot{}) = d()/d\tau$
$\Phi = \cosh(\chi\hat{s}) - \cos(\chi\hat{s}) - \frac{\cosh\chi - \cos\chi}{\sinh\chi - \sin\chi} (\sinh(\chi\hat{s}) - \sin(\chi\hat{s}))$	
$\cos\chi_n \cosh\chi_n = 1, \quad \chi_1 \approx 4.73$	

Table 1: Nondimensional parameters.

3 THEORETICAL PREDICTIONS

3.1 Steady-state model response

Using the method of multiple scales [1], we introduce a uniformly valid expansion $w(t) = w_0(T_0, T_1) + \varepsilon w_1(T_0, T_1) + O(\varepsilon^2)$, where the fast time $T_0 \equiv t$ and the slow time $T_1 \equiv \varepsilon t$ are considered independent, and $\varepsilon \ll 1$. Considering the case of combined direct and parametric primary resonance, i.e. $\Omega = 1 + \varepsilon\sigma$, where σ quantifies the detuning from the fundamental unperturbed natural frequency, and following the standard procedure, the perturbation solution becomes, to first order:

$$w(t) = a \cos(\Omega t - \psi) + \varepsilon \left[\frac{1}{6} \Omega^2 \beta_1 a \cos(2\Omega t + \phi - \psi) + \frac{1}{16} \Omega^2 \beta_2 a \cos(3\Omega t - \psi) - \frac{1}{3} \Omega^2 \eta_2 \cos(2\Omega t) + \frac{1}{32} \kappa_4 a^3 \cos(3\Omega t - \psi) \right] + O(\varepsilon^2), \quad (6)$$

where the steady-state values of the modal amplitude a and phase ψ are solutions of nonlinear algebraic equations:

$$a = \frac{1}{2} \frac{\Omega^2 \eta_1 \sqrt{(\zeta - \frac{1}{4} \Omega^2 \beta_2 \sin(2\phi))^2 + (\Omega - 1 + \frac{1}{4} \Omega^2 \beta_2 \cos(2\phi) - \frac{3}{8} \kappa_4 a^2)^2}}{\left| \zeta^2 - (\frac{1}{4} \Omega^2 \beta_2)^2 + (\Omega - 1 - \frac{3}{8} \kappa_4 a^2)^2 \right|}, \quad (7)$$

$$\psi = \arctan \left(\frac{\frac{1}{4} \Omega^2 \beta_2 \sin(2\phi) - \zeta}{\frac{1}{4} \Omega^2 \beta_2 \cos(2\phi) + \Omega - 1 - \frac{3}{8} \kappa_4 a^2} \right) - \phi. \quad (8)$$

Similar expressions for the amplitude and phase has been derived for a cantilever beam [18], using the lumped-mass model derived by Kumar et al. in [16]. The only difference, as compared to the doubly clamped beam, is in the mode shape functions and the definition on cubic nonlinearities.

For the linearized system ($\kappa_4 = 0$) the steady-state vibration amplitude a is given explicitly by (7), while with nonlinear midplane stretching (7) is a cubic polynomial in a^2 to be solved for a . The resulting steady-state vibration amplitudes for the cantilever beam and the doubly clamped beam are not directly comparable, since the frequency content differ, and the responses of the beams need to be compared at different positions due to their respective mode shapes. To facilitate comparison, we choose to calculate the maximum transverse displacement of both beams, occurring at $\hat{s} = 1$ and $\hat{s} = 1/2$ for the cantilever and doubly clamped beam, respectively. With similar parameters, the transverse displacement will be largest for the cantilever beam; thus, the maximum displacement of the fundamental harmonic ($\varepsilon = 0$ in (6)) for the doubly clamped beam was chosen for reference, and the direct excitation amplitude for the cantilever beam was adjusted accordingly. These results were divided by $\sqrt{2}$ in order to compare

with results obtained by direct numerical integration, for which the RMS values were computed. Computing the RMS values was done to compensate for the possibly asymmetrical steady-state vibration amplitudes and multi-frequency content. It is not required for the present analysis since $\hat{B} = 0$, but nevertheless chosen so to ease comparison with subsequent analyses where the pumping amplitude $\hat{B} > 0$.

Results were multiplied with the beams respective mode shape functions, in turn yielding the frequency responses for displacements as shown in Figures 2(a,b). Approximate analytical and numerical results are seen to agree well for both the cantilever beam and doubly clamped beam. Results for the cantilever and doubly clamped beam are plotted separately, to emphasize that their responses are for different resonant frequencies, and still not directly comparable. However, some qualitative conclusions can be made: The nonlinear effects of the cantilever beam are negligible at small displacements (here below 5% of the beam length), i.e. the backbone is practically vertical (Figure 2(a)), whereas midplane stretching effectively reduces the vibration amplitude, and significantly increases the resonant bandwidth, including overhang, of the doubly clamped beam (Figure 2(b)). The effect of nonlinearity on the cantilever beam response, for larger displacements, is shown in the insert in Figure 2(a). For these small displacements, the cantilever beam response is unaffected by a change in slenderness ratio, but for the doubly clamped beam, the vibration amplitude and the resonant bandwidth changes. The beam slenderness ratio is thereby an effective way of adapting the response of parametric amplifiers for different resonant characteristics.

In the perfectly tuned case, i.e. a 2:1 relationship between the parametric and direct excitation, an excitation phase dependency exists, as illustrated in Figures 3(a,b). From Figure 3(a) it appears that the nonlinear and linearized cases are almost identical. This is expected since the cantilever beam operates in its linear range as noted above. A minimum and maximum is observed at $\pi/4$ and $-\pi/4$, respectively, repeating with period π . For the linearized case, these predictions have been identified previously [15]. A symmetrical relationship is observed between the displacement and phase, centered at the maximum or minimum. For the doubly clamped beam, however, Figure 3(b) indicates that an asymmetrical relationship exists between the displacement and phase, and that it is adjustable through the beam slenderness ratio. Increasing the slenderness ratio appears to *reduce* the transverse displacement at the beams midpoint; reflecting that more slender doubly clamped beams have their resonance frequency shifted further away from $\Omega = 1$, cf. Figure 3(b).

3.2 Gain

For calculating gain, it is not needed to consider varying frequency content and beam positions as done previously; these effects cancel each other out. We use the definition of the gain: $G \equiv a_{\text{pumped}}/a_{\text{unpumped}}$, as proposed in [3], i.e. gain is the ratio of steady-state vibration amplitude of the directly and parametrically excited system, to vibration amplitude of the directly excited only system. Increasing the direct excitation only has no effect on the gain, and zero pumping yields a gain of unity. For a cantilever beam it means that the gain can only be adjusted through the parametric excitation, see Figure 4(a). For a doubly clamped beam, however, the gain can be adjusted via the beam slenderness ratio — not only with respect to the magnitude of the gain, but also in terms of sensitivity towards changes in phase, as seen in Figure 4(b). This may be advantageous for e.g. sensors, since one can easily adjust response curves. The authors are currently investigating the relationship between the gain and input phase for increasing pumping amplitudes, and for beam slenderness ratios considerably higher than used in this work; preliminary results indicate qualitatively different behaviour. The theoretical predictions

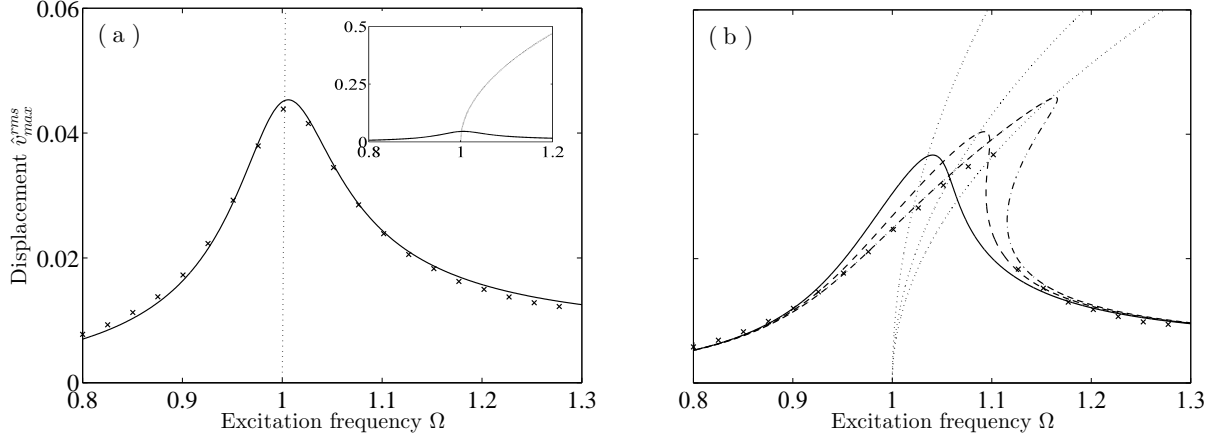


Figure 2: Steady-state vibration displacement $\hat{v}_{max}^{rms}(\hat{s})$ as a function of excitation frequency Ω , obtained by perturbation analysis (lines) and by direct numerical integration (\times) (of (5) for (b) and the similar equation in [18] for (a)): (a) cantilever beam, $\hat{s} = 1$; (b) doubly clamped beam, direct numerical integration (\times) of , $\hat{s} = 1/2$. Beam slenderness ratio: $\lambda = 22$ (—), $\lambda = 31$ (---), and $\lambda = 37$ (-·-·-). For (a) and (b): backbone (·····), $\hat{A} = 0.0058$, $\hat{B} = 0$, $\zeta = 0.05$, $\phi = -\pi/4$, $\alpha = \pi/4$.

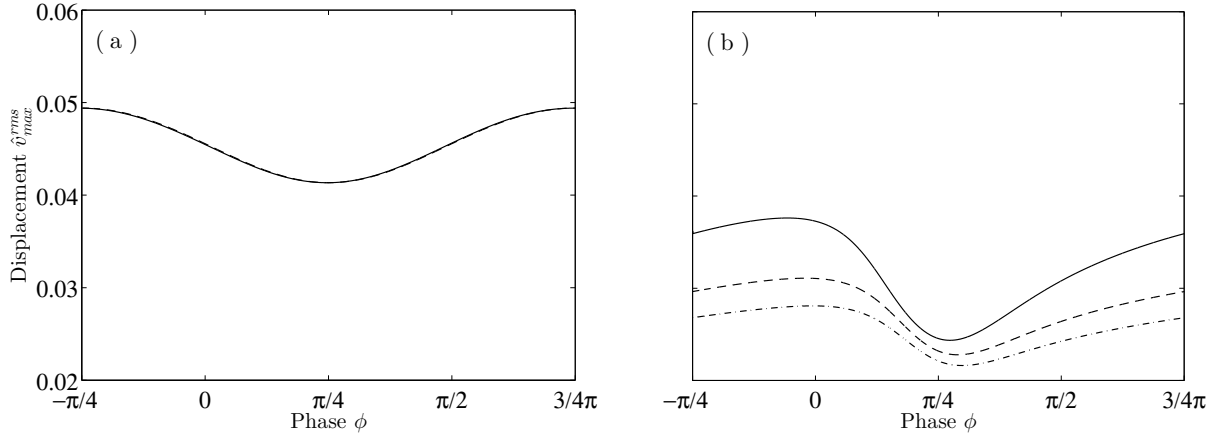


Figure 3: Steady-state vibration displacement $\hat{v}_{max}^{rms}(\hat{s})$ as a function of phase ϕ , obtained by perturbation analysis for: (a) cantilever beam. Nonlinear (—) and linearized (---); (b) doubly clamped beam. For (a) and (b): $\hat{B} = 0.004$, $\Omega = 1$; other parameters as for Figure 2.

are also tested experimentally.

4 CONCLUSIONS

We compared theoretically the effect of specific cubic nonlinearities on the parametrically amplified vibration amplitude and gain. An analytical expression for the vibration amplitude was derived. For a given small level of transverse displacement for both the cantilever and doubly clamped beam, the effect of midplane stretching is dominant, compared to those caused by nonlinear inertia and curvature. For this level of transverse displacement, the cantilever beam effectively operates in its linear regime. For a doubly clamped beam, it was found that the slenderness ratio can sensitively change the output amplitude and gain, and thus be used as an effective design parameter for parametric amplifiers.

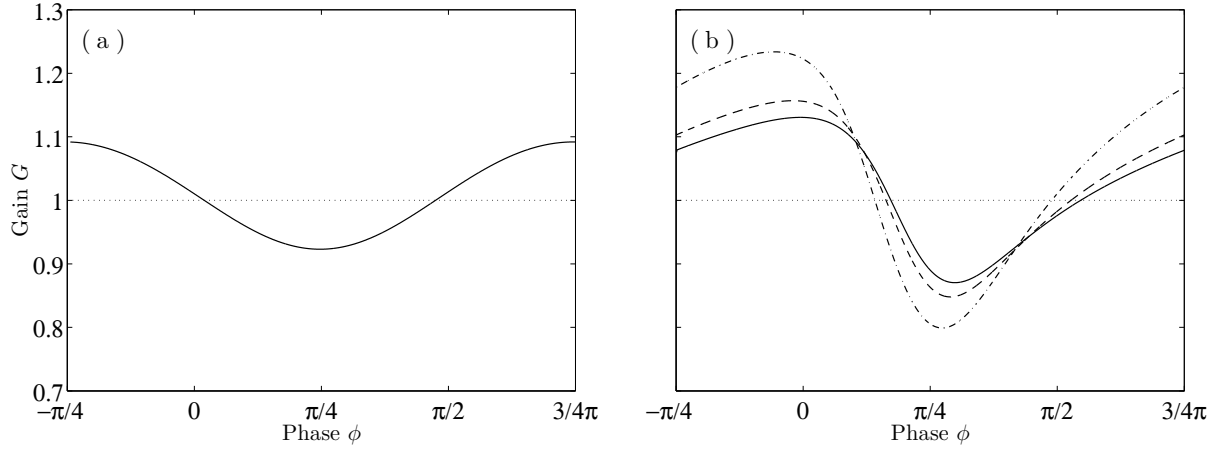


Figure 4: Gain as a function of phase ϕ , obtained by perturbation analysis for: (a) cantilever beam; (b) doubly clamped beam. For (a) and (b): $\Omega = 1$, $\hat{B} = 0.004$ except $\hat{B} = 0$ (·····); other parameters as for Figure 2.

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